Generic initial ideals, sectional matrices and free hyperplane arrangements (Computer Algebra -- Theory and its Applications)

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Abstract

We describe two new characterizations of the freeness for a hyperplane arrangement in terms of the generic initial ideal and of the sectional matrix of the Jacobian ideal of the arrangement.

1 Introduction

An arrangement of hyperplanes is a finite collection of affine subspaces of codimension one in a finite dimensional vector space. Associated to these spaces, there are many algebraic, combinatorial and topological invariants. Arrangements are easily defined but they lead to deep and beautiful results connecting various area of mathematics. We refer to [9] for a comprehensive treatment of this subject.

In the theory of hyperplane arrangements, the freeness of an arrangement is a key notion which connects arrangement theory with algebraic geometry and combinatorics. The notion of freeness was introduced by Saito in [11] for the case of hypersurfaces in the analytic category. The special case of hyperplane arrangements was firstly studied by Terao in [12], where he showed that we can pass from analytic to algebraic considerations. By definition, an arrangement is free if and only if its module of logarithmic derivations is a free module. It turns out that, by Terao's characterization [9], this notion is equivalent to the fact that the Jacobian ideal of the arrangement is Cohen-Macaulay of codimension 2. There are several ways to prove freeness, e.g. using Saito's criterion [11], addition-deletion theorem [12], etc. However, it is not always easy to characterize freeness or to construct new free arrangements.

We will give new characterizations of freeness for any dimension. Namely, we will characterize freeness in terms of the generic initial ideal and of the sectional matrix of the Jacobian ideal of an arrangement.

These results are part of [5]. All the computations in the paper are done using the computer algebra system CoCoA, see [1], [2], [3] and [10].

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2 Preliminares on hyperplane arrangements

Let K be a field of characteristic zero. A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \ldots, H_n\}$ in K^l is called a **hyperplane arrangement**. For each hyperplane H_i we fix a polynomial $\alpha_i \in S = S^*(K^l) = K[x_1, \ldots, x_l]$ such that $H_i = \alpha_i^{-1}(0)$, and let $Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$. An arrangement $\mathcal{A}$ is called **central** if each H_i contains the origin of K^l, and it is called **essential** if there exist $H_{i_1}, \ldots, H_{i_l} \in \mathcal{A}$ such that $\dim(H_{i_1} \cap \cdots \cap H_{i_l}) = 0$.

We denote by $\mathrm{Der}_{K^l} = \{\sum_{i=1}^l f_i \partial_{x_i} \mid f_i \in S\}$ the S-module of **polynomial vector fields** on K^l (or S-derivations). Let $\delta = \sum_{i=1}^l f_i \partial_{x_i} \in \mathrm{Der}_{K^l}$. Then δ is said to be **homogeneous of polynomial degree** d if $f_1, \ldots, f_l$ are homogeneous polynomials of degree d in S, and we write $\mathrm{pdeg}(\delta) = d$.

A central arrangement $\mathcal{A}$ is said to be **free with exponents** $(e_1, \ldots, e_l)$ if and only if the module of vector fields logarithmic tangent to $\mathcal{A}$, that is

$$D(\mathcal{A}) = \{\delta \in \mathrm{Der}_{K^l} \mid \delta(\alpha_i) \in \langle \alpha_i \rangle S, \forall i\},$$

is a free S-module and there exists a basis $\delta_1, \ldots, \delta_l \in D(\mathcal{A})$ such that $\mathrm{pdeg}(\delta_i) = e_i$, or equivalently $D(\mathcal{A}) \cong \bigoplus_{i=1}^l S(-e_i)$. When we say that $\mathcal{A}$ is free with exponents $(e_1, \ldots, e_l)$, we suppose $e_1 \leq \cdots \leq e_l$, and if $\mathcal{A}$ is essential then $e_1 = 1$.

The module $D(\mathcal{A})$ is a graded S-module and we have that $D(\mathcal{A}) = \{\delta \in \mathrm{Der}_{K^l} \mid \delta(Q(\mathcal{A})) \in \langle Q(\mathcal{A}) \rangle S\}$. In particular, since the arrangement $\mathcal{A}$ is central, then the Euler vector field $\delta_E = \sum_{i=1}^l x_i \partial_{x_i}$ belongs to $D(\mathcal{A})$. In this case, $D(\mathcal{A}) \cong S \cdot \delta_E \oplus D_0(\mathcal{A})$, where $D_0(\mathcal{A}) = \{\delta \in \mathrm{Der}_{K^l} \mid \delta(Q(\mathcal{A})) = 0\}$.

Given an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in K^l, the **Jacobian ideal** $J(\mathcal{A})$ of $\mathcal{A}$ is the ideal of S generated by $Q(\mathcal{A})$ and all its partial derivatives.

In [12], Terao proved the following statement.

Theorem 2.1 (Terao's criterion)
A central arrangement $\mathcal{A}$ is free if and only if $S/J(\mathcal{A})$ is 0 or Cohen-Macaulay.

Given an arrangement $\mathcal{A}$, we can compute the minimal resolution of the Jacobian ideal, and if $\mathcal{A}$ is free this is quite easy. See [12, p.296] for more details. Specifically, if $\mathcal{A} = \{H_1, \ldots, H_n\}$ is a central, essential and free hyperplane arrangement with exponents $(1, e_2, \ldots, e_l)$, then $S/J(\mathcal{A})$ has a minimal free resolution of the type

$$0 \to \bigoplus_{i=2}^{l} S(-n - e_i + 1) \cong D_0(\mathcal{A}) \to S(-n+1)^l \to S.$$

Example 2.2
(i) Consider the arrangement $\mathcal{A}$ in $\mathbb{C}^3$ defined by the polynomial $Q(\mathcal{A}) = xyz(x+y)(x-y) \in \mathbb{C}[x,y,z]$. This is a free arrangement with exponents $(1,1,3)$, in fact $S/J(\mathcal{A})$ has resolution

$$0 \to S(-5) \oplus S(-7) \to S(-4)^3 \to S.$$

(ii) Consider the arrangement $\mathcal{A}$ in $\mathbb{C}^3$ defined by the polynomial $Q(\mathcal{A}) = x(x+y-z)(x+z)(x+2z)(x+y+z) \in \mathbb{C}[x,y,z]$. This arrangement is not free, in fact $S/J(\mathcal{A})$ has resolution

$$0 \to S(-8) \to S(-6) \oplus S(-7)^2 \to S(-4)^3 \to S.$$

3 Generic initial ideals and free hyperplane arrangements

We firstly characterize freeness of hyperplane arrangements by looking at the generic initial ideal $\mathrm{rgin}(J(\mathcal{A}))$ of the Jacobian ideal $J(\mathcal{A})$ of $\mathcal{A}$ with respect to the term ordering DegRevLex. For more details and additional properties of generic initial ideals, we refer to Galligo [7] and the book [8].

Definition 3.3
A monomial ideal I in $K[x_1, \dots, x_l]$ is said to be ***strongly stable*** *if for every power-product $t \in I$ and every i, j such that $i < j$ and $x_j | t$, the power-product $x_i \cdot t/x_j$ is in I.*

Theorem 3.4 (Galligo)
Let I be a homogeneous ideal in $K[x_1, \dots, x_l]$ and σ a term ordering such that $x_1 >_\sigma x_2 >_\sigma \dots >_\sigma x_l$. Then there exists a Zariski open set $U \subseteq \mathrm{GL}(l)$ and a strongly stable ideal B such that for each $g \in U$, $\mathrm{LT}_\sigma(g(I)) = B$.

Definition 3.5
The strongly stable ideal B given in Theorem 3.4 is called the ***generic initial ideal with respect to $\boldsymbol{\sigma}$*** *of I and it is denoted by $\mathrm{gin}_\sigma(I)$. In particular, when $\sigma =$ DegRevLex, $\mathrm{gin}_\sigma(I)$ is simply denoted with $\mathrm{rgin}(I)$.*

We are now ready to present our first characterization.

Theorem 3.6
Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement in K^l. Then $\mathcal{A}$ is free if and only if $\mathrm{rgin}(J(\mathcal{A}))$ is S or its minimal generators include x_1^{n-1}, some positive power of x_2, and no monomials in $x_3, \dots, x_l$. More precisely, $\mathcal{A}$ is free if and only if $\mathrm{rgin}(J(\mathcal{A}))$ is S or it is minimally generated by

$$x_1^{n-1},\ x_1^{n-2}x_2^{\lambda_1},\ \dots,\ x_2^{\lambda_{n-1}}$$

with $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1}$ and $\lambda_{i+1} - \lambda_i = 1$ or 2.

Example 3.7
(i) Consider the free arrangement $\mathcal{A}$ in $\mathbb{C}^3$ of the Example 2.2.(i) defined by the polynomial $Q(\mathcal{A}) = xyz(x+y)(x-y) \in \mathbb{C}[x, y, z]$. Then the generic initial ideal of its Jacobian ideal is

$$\mathrm{rgin}(J(\mathcal{A})) = (x^4, x^3y, x^2y^2, xy^4, y^6),$$

as expected by the previous theorem.

(ii) Consider the non-free arrangement $\mathcal{A}$ in $\mathbb{C}^3$ of the Example 2.2.(ii) defined by the polynomial $Q(\mathcal{A}) = x(x+y-z)(x+z)(x+2z)(x+y+z) \in \mathbb{C}[x, y, z]$. Then the generic initial ideal of its Jacobian ideal is

$$\mathrm{rgin}(J(\mathcal{A})) = (x^4, x^3y, x^2y^2, xy^4, y^5, \underline{xy^3z^2}),$$

as prescribed by the previous theorem.

If we look at the resolution of the $\mathrm{rgin}(J(\mathcal{A}))$, we can not only understand if $\mathcal{A}$ is free but we can also compute its exponents.

Theorem 3.8
Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential and central arrangement in K^l, with $l \geq 2$. If $\mathcal{A}$ is free with exponents $(e_1, \dots, e_l)$ then $\mathrm{rgin}(J(\mathcal{A}))$ has free resolution

$$0 \to \bigoplus_{j=n-1}^{n+e_l-2} S(-j-1)^{\beta_{1,j+1}} \to \bigoplus_{j=n-1}^{n+e_l-2} S(-j)^{\beta_{0,j}} \to \mathrm{rgin}(J(\mathcal{A})) \to 0,$$

where $\beta_{0,n-1} = \beta_{1,n}+1 = l$ and $\beta_{1,j+1} = \beta_{0,j} = \#\{i \mid e_i > j-n+1\}$ for all $j \geq n$. In particular, $\beta_{0,n-1} > \beta_{0,n} \geq \cdots \geq \beta_{0,n+e_l-2}$.

Example 3.9
Consider the essential arrangement $\mathcal{A}$ in $\mathbb{C}^3$ of the Example 2.2.(i). $\mathcal{A}$ is free with exponents $(1, 1, 3)$, and $S/J(\mathcal{A})$ has a resolution

$$0 \to S(-5) \oplus S(-7) \to S(-4)^3 \to S.$$

The graded Betti numbers of $S/J(\mathcal{A})$ are bounded by those of $S/\mathrm{rgin}(J(\mathcal{A}))$ which can be computed by the exponents of $\mathcal{A}$ (Theorem 3.8).

$$\beta_{0,4} = 3, \beta_{1,5} = \beta_{0,4} - 1 = 2,$$

$$\beta_{1,6} = \beta_{0,5} = \#\{i \mid e_i > 1\} = \#\{e_3\} = 1,$$
$$\beta_{1,7} = \beta_{0,6} = \#\{i \mid e_i > 2\} = \#\{e_3\} = 1.$$

Thus, the resolution of $S/\mathrm{rgin}(J(\mathcal{A}))$ is

$$0 \to S(-5)^2 \oplus S(-6) \oplus S(-7) \to S(-4)^3 \oplus S(-5) \oplus S(-6) \to S.$$

Corollary 3.10
Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central and essential arrangement in K^l, with $n \geq 2$. If $\mathcal{A}$ is free, then $\mathrm{rgin}(J(\mathcal{A}))$ is uniquely determined by the exponents of $\mathcal{A}$. Vice versa, the exponents of $\mathcal{A}$ are uniquely determined by $\mathrm{rgin}(J(\mathcal{A}))$.

Corollary 3.11
Let $\mathcal{A}$ and $\mathcal{A}'$ be two essential, central and free arrangements. Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are combinatorially equivalent, i.e. they have isomorphic intersection lattices, then $\mathrm{rgin}(J(\mathcal{A})) = \mathrm{rgin}(J(\mathcal{A}'))$.

4 Sectional matrices and free hyperplane arrangements

We now characterize freeness of hyperplane arrangements by looking at the sectional matrix of $S/J(\mathcal{A})$. In this setting, the sectional matrix $\mathcal{M}_{S/I}$ of a polynomial ideal I encodes the Hilbert functions of successive hyperplane sections of the quotient S/I. For more details and properties of sectional matrices, we refer to [6] and [4].

Definition 4.12
Given a homogeneous ideal I in S, the ***sectional matrix*** *of S/I is the function $\{1, \ldots, l\} \times \mathbb{N} \longrightarrow \mathbb{N}$*

$$\mathcal{M}_{S/I}(i, d) = \dim_K(S_d/(I + (L_1, \ldots, L_{l-i}))_d),$$

where $L_1, \ldots, L_{l-i}$ are generic linear forms.

The following result is a rewriting of Lemma 5.5 from [6].

Lemma 4.13
Let I be a homogeneous ideal in S. Then

$$\mathcal{M}_{S/I}(i, d) = \mathcal{M}_{S/\mathrm{rgin}(I)}(i, d) = \dim_K(S_d/(\mathrm{rgin}(I) + (x_{i+1}, \ldots, x_l))_d).$$

Theorem 4.14
Let $\mathcal{A}$ be a central arrangement and $d_0 = \max\{d \mid \mathcal{M}_{S/J(\mathcal{A})}(2, d) \neq 0\}$. Then $\mathcal{A}$ is free if and only if $\mathcal{M}_{S/J(\mathcal{A})}$ is the zero function or the following two conditions hold

1. $\mathcal{M}_{S/J(\mathcal{A})}(3, d_0) = \mathcal{M}_{S/J(\mathcal{A})}(3, d_0{+}1) = \mathcal{M}_{S/J(\mathcal{A})}(3, d_0 + 2)$,
2. $\mathcal{M}_{S/J(\mathcal{A})}(3, d_0) = \sum_{d=0}^{d_0} \mathcal{M}_{S/J(\mathcal{A})}(2, d)$.

Example 4.15
(i) Consider the free arrangement $\mathcal{A}$ in $\mathbb{C}^3$ of the Example 2.2.(i) defined by the polynomial $Q(\mathcal{A}) = xyz(x + y)(x - y) \in \mathbb{C}[x, y, z]$. The sectional matrix of $\mathbb{C}[x, y, z]/J(\mathcal{A})$ is

0	1	2	3	4	$\boxed{5}$	6	7	...
1	1	1	1	0	0	0	0	...
1	2	3	4	2	1	0	0	...
1	3	6	10	12	$\boxed{13}$	$\boxed{13}$	$\boxed{13}$	...

In this case $d_0 = 5$, $\mathcal{M}_{S/J(\mathcal{A})}(3,5) = \mathcal{M}_{S/J(\mathcal{A})}(3,6) = \mathcal{M}_{S/J(\mathcal{A})}(3,7) = 13$ *and* $\mathcal{M}_{S/J(\mathcal{A})}(3,5) = \sum_{d=0}^{5} \mathcal{M}_{S/J(\mathcal{A})}(2,d)$.

(ii) Consider the non-free arrangement $\mathcal{A}$ *in* $\mathbb{C}^3$ *of the Example 2.2.(ii) defined by the polynomial* $Q(\mathcal{A}) = x(x+y-z)(x+z)(x+2z)(x+y+z) \in \mathbb{C}[x,y,z]$. *The sectional matrix of* $\mathbb{C}[x,y,z]/J(\mathcal{A})$ *is*

$$\begin{array}{cccccccccc} \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \boxed{\scriptstyle 4} & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 & \cdots \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 3 & 4 & 2 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 6 & 10 & \boxed{12} & \boxed{12} & \boxed{11} & 11 & \cdots \end{array}$$

In this case $d_0 = 4$, *but* $12 = \mathcal{M}_{S/J(\mathcal{A})}(3,4) = \mathcal{M}_{S/J(\mathcal{A})}(3,5) > \mathcal{M}_{S/J(\mathcal{A})}(3,6) = 11$. *Notice that* $\mathcal{M}_{S/J(\mathcal{A})}(3,4) = \sum_{d=0}^{4} \mathcal{M}_{S/J(\mathcal{A})}(2,d)$.

With the notation of the previous theorem, d_0 coincides with $\min\{d \mid x_2^{d+1} \in \mathrm{rgin}(J(\mathcal{A}))\}$.

Conjecture 4.16
Let $\mathcal{A}$ *be a central arrangement in* K^l. *If* $\mathrm{rgin}(J(\mathcal{A}))$ *has a minimal generator* T *that involves the third variable of* S, *then* $\deg(T) \geq d_0 + 1$.

If the previous conjecture is true, then the statement of Theorem 4.14 becomes easier, as follows:

Corollary 4.17
Let $\mathcal{A}$ *be a central arrangement. Then* $\mathcal{A}$ *is free if and only if* $\mathcal{M}_{S/J(\mathcal{A})}$ *is the zero function or* $\mathcal{M}_{S/J(\mathcal{A})}(3,d_0) = \mathcal{M}_{S/J(\mathcal{A})}(3,d_0+1) = \mathcal{M}_{S/J(\mathcal{A})}(3,d_0+2)$.

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